

SOLUTIONS TO THE SCHWINGER-DYSON EQUATION FOR THE GLUON AND THEIR IMPLICATIONS FOR CONFINEMENT

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ABSTRACT

Using the Schwinger-Dyson equations the possible infrared behaviour of the gluon propagator is studied. Previous work performed in axial gauges is reviewed, the approximations needed detailed and the difficulties of their justification discussed. We then turn to the Landau gauge and investigate the possibility of a gluon propagator less singular than $1/p^2$ when $p^2 \rightarrow 0$. We find that this infrared softened behaviour of the gluon propagator is inconsistent; only an infrared enhanced gluon, as singular as $1/p^4$ when $p^2 \rightarrow 0$ is consistent with the truncated Schwinger-Dyson equation. The implications for confinement and for the modelling of the Pomeron are discussed.

1 Schwinger-Dyson equation approach to QCD

The complex of Schwinger-Dyson equations (SDE) provides a non-perturbative framework in which we can study the infrared properties of QCD and confinement in particular. (For a comprehensive review see Roberts and Williams [1].) Here we describe the infrared behaviour of possible self-consistent solutions of the truncated SDE for the gluon propagator.

The SDE are the field equations of a quantum field theory, inter-relating the n -point Green's functions. They contain all the information of the theory but are impossible to solve exactly since they are an infinite tower of coupled, non-linear integral equations. In QCD the SDE for the gluon, shown diagrammatically in Fig. 1, gives a relation for the propagator in terms of the full 3- and 4-point vertex functions, the quark and ghost propagators and their couplings. Here we consider a world without quarks, i.e. we neglect the quark loop in the SDE. This is

$$\begin{aligned}
& \text{Gluon line with self-energy} = \text{Bare gluon line} + \frac{1}{2} \text{Loop with 2 vertices} \\
& + \frac{1}{2} \text{Loop with 2 vertices} + \frac{1}{6} \text{Loop with 3 vertices} + \frac{1}{2} \text{Loop with 4 vertices} \\
& - \text{Loop with 2 vertices} - \text{Loop with ghost and 2 vertices}
\end{aligned}$$

Figure 1: The Schwinger-Dyson equation for the gluon propagator
Here the broken line represents the ghost propagator

reasonable since we expect the non-Abelian nature of QCD to be responsible for confinement.

The infrared behaviour of the gluon propagator has important implications for quark confinement. A gluon propagator which is as singular as $1/p^4$ when $p^2 \rightarrow 0$ yields a linearly rising interquark potential. Furthermore, West [2] proved, that if, in any gauge, $\Delta_{\mu\nu}$ is as singular as $1/p^4$ then the Wilson operator satisfies an area law, often regarded as a signal for confinement. However, another sufficient condition for confinement is the absence of a pole in the propagator at timelike momenta [3]. So, a gluon propagator which is less singular than $1/p^2$ when $p^2 \rightarrow 0$ describes a confined particle. Such a behaviour was first assumed by Landshoff and Nachtmann [4] in their model of the Pomeron.

Extensive work has been done, investigating the infrared behaviour of the gluon propagator. The different solutions obtained from SDE studies are displayed diagrammatically in Fig. 2. A number of lattice studies have also been performed, but the infrared behaviour of propagators are seriously affected by the lattice's finite size.

The infrared vanishing propagator has been proposed by Stingl et al. [5] using a different method to solve the approximated set of SDEs than the one usually employed. However a study of the quark-SDE using this gluon propagator, [6] and [7], has shown that the infrared vanishing gluon propagator does not support dynamical chiral symmetry breaking and confinement. Thus the full gluon propagator in QCD cannot have this behaviour.

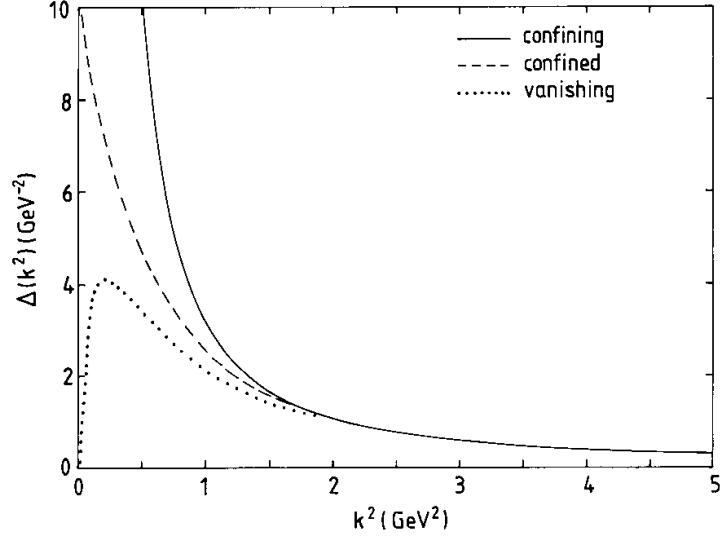


Figure 2: Possible behaviour of the gluon propagator $\Delta(p^2)$

We concentrate therefore on the other two solutions: the infrared enhanced, *confining* solution and the *confined* solution, which has a singularity softer than a pole. A *confined* gluon has only been claimed to exist in the axial gauge [8], whereas a solution as singular as $1/p^4$ has been shown to exist in both axial and Landau gauges [9] - [12].

2 Axial gauge studies

Studies of the axial gauge Schwinger-Dyson equation have the advantage that ghost fields are absent. However the gluon propagator is more complicated than in covariant gauges:

$$\Delta_{\mu\nu}(p^2, \gamma) = -\frac{i}{p^2} \left[F(p^2, \gamma) M_{\mu\nu} + H(p^2, \gamma) N_{\mu\nu} \right] \quad , \quad (1)$$

with the tensors given by:

$$M_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{n \cdot p} + n^2 \frac{p_\mu p_\nu}{(n \cdot p)^2} \quad , \quad \text{and} \quad N_{\mu\nu} = g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2} \quad ,$$

and $\gamma = (n \cdot p)^2 / (n^2 p^2)$ is the axial gauge parameter. However in all previous studies it is assumed that the full propagator has the same tensor structure as the free one, by setting one of the two axial gauge gluon renormalisation functions, $H(p^2, \gamma)$, to zero. As a consequence all 4-gluon terms can be projected out of the SDE and approximating the 3-gluon vertex by its longitudinal part, which is determined by the Slavnov-Taylor identity one finds a closed integral equation for the inverse of the gluon propagator.

The resulting approximate gluon-SDE has been studied numerically by Baker, Ball and Zachariasen (BBZ) [9] and an infrared enhanced gluon propagator, as singular as $1/p^4$ was found to be a consistent solution. However it has been argued [13] that in axial gauges, in which only positive norm-states occur, a behaviour more singular than $1/p^2$ is not possible and consequently the neglected axial gauge renormalisation function H must cancel any $1/p^4$ singularity in the infrared. More recently, Cudell and Ross [8] have shown that an alternative solution to Schoenmaker's approximation [10] to BBZ's equation exists with an infrared softened gluon propagator. However this solution is found only with an incorrect sign in Schoenmaker's equation. Correcting this error, the axial gauge SDE does not allow an infrared softened solution for the propagator [14].

Because of the difficulty in justifying the neglect of one of the key gluon renormalisation functions in axial gauges, we turn our attention to covariant gauges and the Landau gauge in particular.

3 Landau gauge studies

The advantage of Landau gauge studies is the much simpler structure of the gluon propagator, which is defined by :

$$\Delta_{\mu\nu} = -i \frac{G(p^2)}{p^2} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) . \quad (2)$$

However in any covariant gauge ghosts are necessary to ensure the transversality of the gluon — nothing comes for free! The gluon-SDE is approximated by neglecting all 4-gluon terms, which can be regarded as the first step in a truncation of the SDE, as discussed by Mandelstam [11], and including ghosts only perturbatively. The latter leads to an approximation of the Slavnov-Taylor identity, which is now treated as in the Abelian case. Again we replace the 3-gluon vertex by its longitudinal part determined by this Slavnov-Taylor identity. The resulting truncated Schwinger-Dyson equation is:

$$\begin{aligned} \frac{1}{G(p^2)} = 1 + \frac{g^2 C_A}{96\pi^4} \frac{1}{p^2} \int d^4k \left[G(q^2) A(k^2, p^2) + \frac{G(k^2)G(q^2)}{G(p^2)} B(k^2, p^2) \right. \\ \left. + \frac{G(k^2) - G(p^2)}{k^2 - p^2} \frac{G(q^2)}{G(p^2)} C(k^2, p^2) + \frac{G(q^2) - G(k^2)}{q^2 - k^2} D(k^2, p^2) \right], \end{aligned} \quad (3)$$

with $q = p - k$ and where A , B , C and D are functions of p^2 and k^2 as given in [12].

Brown and Pennington [12] studied this equation numerically and found an infrared singular, *confining* gluon to be a consistent solution. To allow an analytic study of the Brown-Pennington equation, we approximate $G(q^2)$ by $G(p^2 + k^2)$, as first proposed by Schoenmaker [10]. This

should be exact in the infrared limit. Now the angular integrals can be performed analytically giving the following simpler equation:

$$\begin{aligned} \frac{1}{G(p^2)} = 1 + & \frac{g^2 C_A}{48\pi^2} \frac{1}{p^2} \left\{ \int_0^{p^2} dk^2 \left[G_1 \left(-1 - 10 \frac{k^2}{p^2} + 6 \frac{k^4}{p^4} + \frac{k^2}{p^2 - k^2} \left(\frac{75}{4} - \frac{39}{4} \frac{k^2}{p^2} + 4 \frac{k^4}{p^4} - 5 \frac{p^2}{k^2} \right) \right) \right. \right. \\ & + G_2 \left(-\frac{21}{4} \frac{k^2}{p^2} + 7 \frac{k^4}{p^4} - 3 \frac{k^6}{p^6} \right) + G_3 \left(\frac{k^2}{p^2 - k^2} \left(-\frac{27}{8} - \frac{11}{4} \frac{k^2}{p^2} - \frac{15}{8} \frac{p^2}{k^2} \right) \right) \Big] \\ & + \int_{p^2}^{\infty} dk^2 \left[G_1 \left(\frac{p^2}{k^2} - 6 + \frac{p^2}{p^2 - k^2} \left(\frac{29}{4} + \frac{3}{4} \frac{p^2}{k^2} \right) \right) + \right. \\ & \left. \left. + G_2 \left(-\frac{3}{2} + \frac{1}{4} \frac{p^2}{k^2} \right) + G_3 \left(\frac{p^2}{p^2 - k^2} \left(\frac{3}{4} - \frac{67}{8} \frac{p^2}{k^2} - \frac{3}{8} \frac{p^4}{k^4} \right) \right) \right] \right\}, \end{aligned} \quad (4)$$

where

$$G_1 = G(p^2 + k^2), \quad G_2 = G(p^2 + k^2) - G(k^2) \quad \text{and} \quad G_3 = \frac{G(k^2)G(p^2 + k^2)}{G(p^2)}.$$

This equation has a quadratic ultraviolet divergence, which would give a mass to the gluon. Such terms are subtracted to ensure the masslessness condition

$$\lim_{p^2 \rightarrow 0} \frac{1}{\Delta(p^2)} = 0 \quad , \text{ i.e. } \quad \frac{p^2}{G(p^2)} = 0 \quad \text{for } p^2 \rightarrow 0 \quad , \quad (5)$$

is satisfied. This property can be derived generally from the Slavnov-Taylor identity and always has to hold. To determine the possible self-consistent behaviour of the gluon renormalisation function, $G(p^2)$ is expanded in a series in powers of p^2/μ^2 for $p^2 < \mu^2$ (including possible negative powers). Here we describe only the lowest powers for illustration. μ^2 is the mass scale above which we assume perturbation theory applies and we demand that for $p^2 > \mu^2$ the solution of the integral equation matches the perturbative result, i.e. we have $G(p^2) = 1$ modulo logarithms.

To ensure that making Schoenmaker's approximation [10] does not qualitatively alter the behaviour of Eq. (1) we first reproduce the infrared enhanced *confining* gluon found by Brown and Pennington. Taking (cf. [12])

$$G_{in}(k^2) = A \left(\frac{\mu^2}{k^2} \right)_+$$

as an input function, where the $+$ serves as an infrared regulator, we find:

$$\frac{1}{G_{out}(p^2)} = 1 + \frac{\text{const}}{p^2}.$$

This violates the masslessness condition of Eq. (5) and has to be mass renormalised. Then, if terms in $G(p^2)$, of higher order in p^2 , give a contribution to the right hand side of the

equation cancelling the explicit factor of 1, we have the possibility of finding self-consistency. Consequently, we take

$$G_{in}(p^2) = \begin{cases} A(\mu^2/p^2)_+ + (p^2/\mu^2) & \text{if } p^2 < \mu^2 \\ 1 & \text{if } p^2 > \mu^2 \end{cases}, \quad (6)$$

and find after mass renormalisation:

$$\frac{1}{G_{out}(p^2)} = 1 + \frac{g^2 C_A}{48\pi^2} \left[-\frac{479}{24} \frac{p^2}{\mu^2} + \frac{13}{8} \frac{p^2}{\mu^2} \ln \left(\frac{\mu^2}{p^2} \right) - \frac{81}{4} - \frac{25}{4} \ln \left(\frac{\Lambda^2}{\mu^2} \right) \right]. \quad (7)$$

Hence an infrared enhanced *confining* gluon propagator is a self-consistent solution to the SDE. This is the result found by Brown and Pennington [12]. Higher powers of p^2 in Eq. (6) do not affect the answer.

To check whether Eq. (2) allows an infrared softened gluon propagator, i.e. an infrared vanishing renormalisation function, we take (cf. [8])

$$G_{in}(p^2) = \begin{cases} (p^2/\mu^2)^{1-c} & \text{if } p^2 < \mu^2 \\ 1 & \text{if } p^2 > \mu^2 \end{cases} \quad (8)$$

as a trial input function and substitute it into the right hand side of the integral equation, Eq. (2). Performing the k^2 -integration, we obtain, after mass renormalisation:

$$\frac{1}{G_{out}(p^2)} = 1 + \frac{g^2 C_A}{48\pi^2} \left[D_1 + D_2 \left(\frac{\mu^2}{p^2} \right)^{1-c} + D_3 \left(\frac{p^2}{\mu^2} \right)^{1-c} + D_4 \left(\frac{p^2}{\mu^2} \right)^c + \dots \right], \quad (9)$$

where G_1, G_2 and G_3 have been expanded for small p^2 and only the first few terms have been collected in this equation so that

$$D_2 = - \left(\frac{3}{4(2-2c)} + \frac{3}{4} \ln \left(\frac{\Lambda^2}{\mu^2} \right) \right). \quad (10)$$

The other D 's can be found in [14]. Thus the dominant infrared behaviour is:

$$\frac{1}{G_{out}(p^2)} \rightarrow - \left(\frac{\mu^2}{p^2} \right)^{1-c} \quad (11)$$

and self-consistency is spoiled by a negative sign, just as in axial gauges [14]. Note that higher order terms in p^2 in the input function have no qualitative effect. We thus see that an infrared softened *confined* gluon is not possible.

4 Consequences for modelling the Pomeron

How does this infrared behaviour of the gluon affect the Pomeron of Landshoff and Nachtmann [4]? Their belief in an infrared softened, rather than enhanced, gluon rests on their model

requirement that the integral

$$\int_0^\infty dp^2 \Delta(p^2)^2$$

should be finite. However, we do not believe that whether this integral is finite or not is relevant to the finiteness of total cross-sections [15]. The Landshoff-Nachtmann picture imagines that the two dressed gluons modeling their Pomeron couple to single quarks with the other quarks in each initial state hadron being spectators. In this way the forward hadronic scattering amplitude is viewed as quark-quark scattering. The total cross-section is then just the imaginary part of this forward elastic quark scattering amplitude, by the optical theorem. However, an imaginary part is only generated if the quarks can be on mass-shell and have poles in their propagators which is in conflict with confinement. Hadronic amplitudes are not just the result of free quark interactions, perturbative QCD is only valid for hard short distance processes. In soft, non-perturbative physics, the bound state nature of hadrons has to be taken into account.

5 Summary

We have studied the Schwinger-Dyson equation of the gluon propagator in the Landau gauge to determine analytically the possible infrared solutions for the gluon renormalisation function $G(p^2)$. We find only an infrared enhanced gluon, as singular as $1/p^4$ for $p^2 \rightarrow 0$ is consistent with the truncated SDE. This behaviour of the gluon is not at variance with the Pomeron and is in accord with quark confinement. Furthermore it leads to a good phenomenology of hadron observables [1], [16].

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